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Local properties of analytic functions

by

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Local properties of analytic functions

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1. The Weierstrass Preparation Theorem.

(Properties of the local ring A_n).

Definition 1.1. If $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ (or more generally z is in a complex manifold), let $A_{n,z}$ or A_z denote the set of equivalence classes of functions f which are analytic (holomorphic) in some neighborhood of z, under the equivalence relation: $f \sim g$ if $f=g$ in some neighborhood of z. If f is analytic in a neighborhood of z, we write $(f)_z$ for the residue class of f in A_z, which is called the germ of f at z. $A_{n,z}$ is called the ring of germs of analytic functions at the point $z \in \mathbb{C}^n$. Instead of $A_{n,0}$, we also write A_n. For the residue class $(f)_0$ we shall often write f, thus identifying f with its germ at the origin.

It is clear that A_n can be made into a ring with unit. Its elements can be identified with the set of all power series

$$\sum a_\nu z^\nu$$

which converge in some neighborhood of 0, that is, with the set of all arrays $\{a_\nu\}$ such that

$$\sum |a_\nu| r^{|\nu|} < \infty \qquad \text{for some } r>0.$$

The ring A_n is an integral domain; this follows from the identity theorem for analytic functions.
Furthermore, it is clear that $f \in A_n$ has an inverse in A_n iff $f(0) \neq 0$; this follows from the fact that, if f is analytic in a neighborhood of 0 and $f(0) \neq 0$, then $\frac{1}{f(z)}$ is also analytic in a neighborhood of 0. So the nonunits of A_n are precisely the germs of functions which vanish at the origin; and these obviously form an ideal in A_n. The ring A_n is therefore a local ring (i.e. a ring in which the nonunits form an ideal).

For further investigation of the properties of the ring A_n it is convenient to develop a technique which facilitates the induction step from A_{n-1} to A_n. Here we consider A_{n-1} as a subring of A_n (A_{n-1} consists of all germs of analytic functions f(z) which are independent of the variable z_n). The idea is now to effect the transition from A_{n-1} to A_n in two stages by introducing the ring $A_{n-1}[z_n]$.

Definition 1.2. $A_{n-1}[z_n]$ is the polynomial ring over A_{n-1} in the "indeterminate" z_n. So an element of $A_{n-1}[z_n]$ can be written in the form

$$a_0 + a_1 z_n + \ldots + a_k z_n^k$$

where $a_0, a_1, \ldots, a_k \in A_{n-1}$.

Then we have:

$$A_{n-1} \subset A_{n-1}[z_n] \subset A_n \ .$$

The polynomial extension $A_{n-1} \subset A_{n-1}[z_n]$ is handled by means of algebraic theorems (such as Gauss' theorem on unique factorization in a polynomial ring and the Hilbert basis theorem). The extension $A_{n-1}[z_n] \subset A_n$ is treated by the so-called Weierstrass preparation theorem. To formulate this theorem we need some further definitions.

Definition 1.3. An element $f \in A_n$, which is normalized in the z_n-direction (i.e. $f(0,z_n)$ does not vanish identically) is called regular or order p, if there exists a representative function for f (which we call also f) such that $f(0,z_n)$ has a zero of order p at $z_n = 0$. A Weierstrass polynomial π of degree p $(p \geq 1)$ in z_n is an element of $A_{n-1}[z_n]$ of the form

$$\pi = z_n^p + \sum_{\nu=1}^{p} a_\nu z_n^{p-\nu} \quad ,$$

where the coefficients $a_\nu \in A_{n-1}$ are nonunits.

Theorem 1.1. (Weierstrass Preparation Theorem). Let $f \in A_n$ be regular of order p in z_n. Then there exists a unique Weierstrass polynomial $\pi \in A_{n-1}[z_n]$ of degree p such that

$$f = u\pi$$

for some unit $u \in A_n$.
Furthermore, any $g \in A_n$ can be written in a unique manner in the form

$$g = af + b \ ,$$

where $a \in A_n$ and $b \in A_{n-1}[z_n]$ is a polynomial of degree < p.

We shall prove the theorem in a more general form which is expressed in terms of functions, rather than just in terms of germs of functions.

Theorem 1.2. (Weierstrass Preparation Theorem). Let f be analytic in a neighborhood of the closure of the set

$$\Delta = \{z \in \mathbb{C}^n \mid |z_1|<r_1,\ldots,|z_n|<r_n\}$$

and suppose that $f(0,z_n)$ is zero only for $z_n = 0$ in $|z_n| \leq r$, and that the origin is a zero of order p $(p \geq 1)$. Suppose moreover that

$$f(z) \neq 0 \quad \text{for } |z_j| \leq r_j \quad (j \leq n-1),\ |z_n| = r_n.$$

Then, for any function g analytic on Δ, there exist analytic functions

a on Δ', and

$b_1,\ldots,b_p$ on Δ',

where $\Delta' = \{(z_1,\ldots,z_{n-1}) \in \mathbb{C}^{n-1} \mid |z_j|<r_j, j \leq n-1\}$,

such that

(1) $$g = af + \sum_{\nu=1}^{p} b_\nu z_n^{p-\nu} \quad \text{on } \Delta .$$

Also, there exist analytic functions

u on Δ', which is nowhere zero, and

$a_1,\ldots,a_p$ on Δ', $a_\nu(0) = 0 \quad (1 \leq \nu \leq p)$,

such that

(2) $$f = u(z_n^p + \sum_{\nu=1}^{p} a_\nu z_n^{p-\nu}) .$$

Moreover, there exists a constant $M>0$, depending only on Δ and f, such that, in (1) we have

(3) $$\sup_{\Delta} |a| \ , \ \sup_{\Delta'} |b_\nu| \leq M \sup_{\Delta} |g|$$

The representation (1) is unique.

Proof.

We begin by proving the existence of u and a_ν. Let $z' = (z_1,\dots,z_{n-1}) \in \Delta'$, and set

$$\sigma_0(z') = \frac{1}{2\pi i}\int_{|\zeta|=r_n} \frac{\partial f(z',\zeta)}{\partial \zeta}\frac{1}{f(z',\zeta)}\,d\zeta .$$

Then, $\sigma_0(z')$ is the number of zeros of $f(z',z_n)$ in $|z_n|<r_n$; so $\sigma_0(0) = p$. Further, $\sigma_0(z')$ is clearly a continuous function of z'; hence $\sigma_0(z') = p$ for $z'\in\Delta'$. Let $\zeta_1(z'),\dots,\zeta_p(z')$ denote the zeros of $f(z',\zeta)$ in $|\zeta|<r_n$. Then set

$$\pi(z',z_n) = \prod_{j=1}^{p}(z_n - \zeta_j(z')) .$$

We have, for $k\geq 0$,

$$\sigma_k(z') \overset{\text{def}}{=} \sum_{j=1}^{p}\{\zeta_j(z')\}^p = \frac{1}{2\pi i}\int_{|\zeta|=r_n}\frac{\partial f(z',\zeta)}{\partial\zeta}\frac{\zeta^k}{f(z',\zeta)}\,d\zeta ,$$

(cf. [A, p.124])

so that $\sigma_k(z')$ is analytic on Δ'. If

$$a_\nu(z') = (-1)^\nu \sum_{1\leq j_1<\dots<j_\nu\leq p} \zeta_{j_1}(z')\dots\zeta_{j_p}(z')$$

is the ν-th elementary symmetric functions of the ζ_j, then a_ν is a polynomial in $\sigma_1,\dots,\sigma_p$ (with rational coefficients), so that a_ν is analytic on Δ', and

$$\pi(z',z_n) = z_n^p + \sum_{\nu=1}^{p} a_\nu(z')z_n^{p-\nu} .$$

For any $z'\in\Delta'$, f and π have the same zeros. Hence

$$\frac{f(z',z_n)}{\pi(z',z_n)} = \frac{1}{2\pi i}\int_{|\zeta|=r_n}\frac{f(z',\zeta)}{\pi(z',\zeta)}\frac{1}{\zeta-z_n}\,d\zeta$$

is analytic on Δ. Similarly for $\frac{\pi(z',z_n)}{f(z',z_n)}$.

Hence $f = u\pi$ where u is analytic and nowhere zero on Δ;
Further, it is clear that we may replace Δ by a slightly larger polydisc. without altering the conditions of the theorem. Hence u and u^{-1} are bounded on Δ, and so are the coefficients of π.

To prove the division algorithm and inequality (3) we may replace f by π. If g is analytic on Δ, let

$$a(z',z_n) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\frac{1}{\zeta-z_n}\,d\zeta ,$$

where $0<r<r_n$. Clearly the integral is independent of r if $|z_n|<r$, so that a is analytic on Δ. Then, if $|z_n|<r$,

$$g(z',z_n) - a(z',z_n)\pi(z',z_n) = \frac{1}{2\pi i}\int_{|\zeta|=r} g(z',\zeta)\left\{1 - \frac{\pi(z',z_n)}{\pi(z',\zeta)}\right\}\frac{1}{\zeta-z_n}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\left\{\frac{\zeta^p - z_n^p + \sum\limits_{\nu=1}^{p} a_\nu(z')\,(\zeta^{p-\nu} - z_n^{p-\nu})}{\zeta - z_n}\right\}d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\sum_{\nu=1}^{p} c_\nu(z',\zeta)\quad z_n^{p-\nu}\,d\zeta ,$$

where $c_\nu(z',\zeta)$ is a polynomial in ζ of degree $< p$ whose coefficients are linear combinations of the a_ν. If we set

$$b_\nu(z') = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{g(z',\zeta)}{\pi(z',\zeta)}\,c_\nu(z',\zeta)\,d\zeta ,$$

we obtain

$$g(z) - a(z)\pi(z) = \sum_{\pi=1}^{p} b_\nu(z')\,z_n^{p-\nu} .$$

Moreover the a_ν are bounded on Δ and $\pi(z',\zeta)$ is bounded away from zero on the set $|z_j|\le r_j$, $j\le n-1$, $|\zeta| = r_n$. Hence

$$\sup_{\Delta'} |b_\nu| \leq M \sup_{\Delta'} |g| ,$$

where M depends only on the a_ν and π. Hence, since

$$g - a\pi = \sum_{=1}^{p} b_\nu z_n^{p-\nu} ,$$

we deduce that for fixed $z' \in \Delta'$, we have

$$|a(z',z_n)| \leq \frac{M' \sup_{\Delta} |g|}{\inf_{|\zeta|=r_n} |\pi(z',\zeta)|} ,$$

so that

$$\limsup_{|z_n|\to r_n} |a(z',z_n)| \leq M'' \sup_{\Delta} |g| .$$

By the maximum principle, this implies that

$$\sup_{\Delta} |a| \leq \sup_{\Delta} |g| .$$

To demonstrate the uniqueness of (1), suppose that we have two such expressions:

$$g = af + \sum_{\nu=1}^{p} b_\nu z_n^{p-\nu} = a'f + \sum_{\nu=1}^{p} b'_\nu z_n^{p-\nu} .$$

Then

$$(a - a')f = \sum_{\nu=1}^{p} (b'_\nu - b_\nu) z_n^{p-\nu} .$$

This implies that for fixed $z' \in \Delta'$, the polynomial

$$\sum_{\nu=1}^{p} (b'_\nu - b_\nu) z_n^{p-\nu}$$

of degree $\leq p-1$ has at least p zeros in $|z_n|<r_n$; this implies $b_\nu = b'_\nu$ and it follows that $a = a'$.

This completes the proof Theorem 1.2.

As an application of the Weierstrass preparation theorem, we shall derive some further properties of the local ring A_n.

Definition 1.4. An element $f \in A_n$ (resp. $A_{n-1}[z_n]$) is called reducible over A_n (resp. $A_{n-1}[z_n]$) if it can be written as a product $f = g_1 g_2$, where g_1, g_2 are nonunits of A_n (resp. $A_{n-1}[z_n]$); elements without this property are called irreducible over A_n (resp. $A_{n-1}[z_n]$). A unique factorization domain is an integral domain with an identity element in which every nonunit can be written as a finite product of irreducible factors, and in which such a factorization is unique up to the order of its factors and units of the ring.

Lemma 1.1. Let $f = g\pi$, where f, g and $\pi \in A_n$. If $f \in A_{n-1}[z_n]$ and if $\pi \in A_{n-1}[z_n]$ is a Weierstrass polynomial, then $g \in A_{n-1}[z_n]$.
Proof. Since π is monic (the leading coefficient of π, as a polynomial in z_n, is equal to 1), we can make an algebraic division over A_{n-1}; then we obtain

$$f = g'\pi + r,$$

where g' and r belong to $A_{n-1}[z_n]$ and r is a polynomial of degree lower than that of π. But the uniqueness stated in the Weierstrass preparation theorem then implies $g' = g$ and $r = 0$.

Lemma 1.2. A Weierstrass polynomial $\pi \in A_{n-1}[z_n]$ is reducible over A_n iff it is reducible over $A_{n-1}[z_n]$. If π is reducible, then all of its factors are Weierstrass polynomials, modulo units of $A_{n-1}[z_n]$.
Proof. Suppose first $\pi = g_1 g_2$ where $g_j \in A_n$ are nonunits; since π is a Weierstrass polynomial, both g_1 and g_2 are regular in z_n. Applying the Weierstrass preparation theorem, write $g_j = u_j \pi_j$ $(j = 1,2)$, where $u_j \in A_n$ are units and $\pi_j \in A_{n-1}[z_n]$ are Weierstrass polynomials; thus $\pi = (u_1 u_2)(\pi_1 \pi_2)$. From the uniqueness it follows that $u_1 u_2 = 1$ and $\pi = \pi_1 \pi_2$.
Second, suppose that $\pi = g_1 g_2$ where $g_j \in A_{n-1}[z_n]$ are nonunits of that ring. If g_1 were a unit in A_n, then $g_2 = \frac{1}{g_1}\pi$. From lemma 1.1. it would follow that $\frac{1}{g_1} \in A_{n-1}[z_n]$; this is impossible, since g_1 is a nonunit of $A_{n-1}[z_n]$, and therefore g_1 is a nonunit of A_n.

Theorem 1.3. The local ring A_n is an unique factorization domain.
Proof. When $n = 0$ the theorem is trivial. Therefore assume that the theorem holds for A_{n-1}. We apply now Gauss' theorem:

> If A is an unique factorization domain, then so is the polynomial ring $A[X]$ (cf. [W 1,p.70]).

So it follows that $A_{n-1}[z_n]$ is also an unique factorization domain. Consider any element $f \in A_n$; by a suitable linear change of coordinates we can make f regular in z_n. Then, by the Weierstrass preparation theorem, write $f = u\pi$ where $u \in A_n$ is an unit, $\pi \in A_{n-1}[z_n]$ is a Weierstrass polynomial. The polynomial π can be written uniquely, up to order and units in $A_{n-1}[z_n]$, as a product of irreducible polynomials; this provides in view of lemma 1.2. an unique factorization in A_n, up to order and units in A_n.

Definition 1.5. A commutative ring A with an identity element is called Noetherian if every ideal $I \subset A$ is finitely generated over A, that is, if there exist elements $f_1, \ldots, f_k \in I$, so that every $f \in I$ can be written

$$f = \sum_{i=1}^{k} a_i f_i$$

for some $a_i \in A$.

Theorem 1.4. The local ring A_n is Noetherian.
Proof. For $n = 0$ the theorem is trivial. Assume that the theorem has already been proved for A_{n-1}. We apply the Hilbert basis theorem:

> If A is a Noetherian ring, then so is the polynomial ring $A[X]$ (cf. [W 2,p.18]).

So $A_{n-1}[z_n]$ is a Noetherian ring. Consider any ideal $I \subset A_n$ which contains some non zero elements f. After a linear change of coordinates we may suppose that f is regular in z_n of order p. After multiplication by an unit we can further assume that $f \in I \cap A_{n-1}[z_n]$ is a Weierstrass polynomial. Since $A_{n-1}[z_n]$ is Noetherian the ideal $I \cap A_{n-1}[z_n]$ has a finite set of generators. Now $f, f_1, \ldots, f_k$ generate the entire ideal I, which we see as follows. If $g \in I$, write $g = af + b$, where $b \in A_{n-1}[z_n]$; but b is clearly also in I, hence in $I \cap A_{n-1}[z_n]$, so that

$$b = \sum_{i=1}^{k} a_i f_i$$

for some elements $a_i \in A_{n-1}[z_n]$. Thus

$$g = af + \sum_{i=1}^{k} a_i f_i \ .$$

2. The Oka Theorem.

(Coherence of the sheaf $\mathcal{A}_{n,\Omega}$).

Up to now we have only considered the ring $A_{n,0}$, that is A_z for fixed $z \in \mathbb{C}^n$. In the present section we let z be variable; we shall prove a theorem (the Oka theorem) which goes beyond the Noetherian property of A_z.

Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{A}_{n,\Omega}$ the sheaf on Ω of germes of analytic functions.

Definition 2.1. Let $\mathcal{F}$ be an analytic sheaf over Ω (i.e. a sheaf of $\mathcal{A}$-modules over Ω). $\mathcal{F}$ is said to be locally finitely generated if for every $z \in \Omega$ there exists a neighborhood $U \subset \Omega$ of z and a finite number of sections $f_1, \dots, f_q \in \Gamma(U, \mathcal{F})$ so that $\mathcal{F}_\zeta$ is generated by $(f_1)_\zeta, \dots, (f_q)_\zeta$ as an A_ζ-module for every $\zeta \in U$.

Definition 2.2. Let $\mathcal{F}$ be an analytic sheaf over Ω. Let $f_1, \dots, f_q \in \Gamma(U, \mathcal{F})$ where $U \subset \Omega$ is an open set and let $z \in U$. If there exists a tuple $(\alpha_1, \dots, \alpha_q) \in A_z^q$ such that

$$\sum_{i=1}^{q} \alpha_i (f_i)_z = 0,$$

the tuple $(\alpha_1, \dots, \alpha_q)$ is called a relation between $f_1, \dots, f_q$ at z. The collection of all such relations forms an analytic sheaf over Ω (it is a subsheaf of $\mathcal{A}^p_{n,\Omega}$ since it is contained in $\mathcal{A}^p_{n,\Omega}$ as an open set). It is denoted by $\mathcal{R}(f_1, \dots, f_q)$ and it is called the sheaf of relations between $f_1, \dots, f_q$.

Definition 2.3. An analytic sheaf $\mathcal{F}$ over Ω is called coherent if

(i) $\mathcal{F}$ is locally finitely generated,

(ii) if $U \subset \Omega$ is an open set and $f_1, \dots, f_q \in \Gamma(U, \mathcal{F})$ then the sheaf of relations $\mathcal{R}(f_1, \dots, f_q)$ is locally finitely generated.

Lemma 2.1. A subsheaf $\mathcal{G}$ of a coherent sheaf $\mathcal{F}$ over Ω is coherent iff it is locally finitely generated.
Proof. Any section of $\mathcal{G}$ is a section of $\mathcal{F}$. Hence the sheaf of relations $\mathcal{R}$ of any finite number of sections of $\mathcal{G}$ is the sheaf of relations between these sections, considered as sections of $\mathcal{F}$. Since $\mathcal{F}$ is coherent, $\mathcal{R}$ is locally finitely generated.

We now come to our main theorem.

Theorem 2.1. The sheaf $\mathcal{A}^p$ (= $\mathcal{A}^p_{n,\Omega}$) is a coherent sheaf of rings.
Proof. $\mathcal{A}^p$ is locally finitely generated since the sections E_j, $1 \leq j \leq p$, defined by

$$E_j = (0,\ldots,0,1,0,\ldots,0) \quad \text{(1 in the j-th place)}$$

generate the stalk $\mathcal{A}^p_z$ at each point $z \in \Omega$. Hence the theorem reduces to the following one.

Theorem 2.2. (Oka Theorem). Let $F_1,\ldots,F_q \in A(\Omega)^p$ (= $\Gamma(\Omega,\mathcal{A}^p)$), and let $\mathcal{R}$ be the sheaf of relations between $F_1,\ldots,F_q$. Then $\mathcal{R}$ is locally finitely generated.

Remark. Since A_z is Noetherian, we know of course already that $\mathcal{R}_z$ is finitely generated for every $z \in \Omega$, but the important point in the theorem is that one can use "the same" generators for all ζ in a neighborhood of any given point z.

Proof. The proof consists of two parts.
(A) First we prove the theorem for $p>1$ assuming that it has already been proved for smaller values of p.
(B) Second we prove the case $p = 1$ assuming that the theorem has already been proved for every p in the $(n-1)$-dimensional case.
The theorem follows from (A) and (B).

(A)
Assume $z = 0 \in \Omega$. Then we have to construct a neighborhood U of 0 with the properties stated in Def. 2.1.

We shall use the following notations.
Let

$$F_1 = \begin{pmatrix} f_1^1 \\ \vdots \\ f_1^p \end{pmatrix}, \ \dots \ , F_q = \begin{pmatrix} f_q^1 \\ \vdots \\ f_q^p \end{pmatrix}$$

belong to $A(\Omega)^p$. The matrix with columns $F_1,\dots,F_q$ is denoted by F. Instead of $\mathcal{R}(F_1,\dots,F_q)$ we write also $\mathcal{R}(F)$.
Let

$$\alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} \in A_z^q \quad ;$$

then the element of $A_z{}^q$ that we obtain by applying the matrix $(F)_z$ (this is the matrix with columns $(F_1)_z,\dots,(F_q)_z$) to the vector α is denoted by

$$(F)_z\alpha \quad .$$

Then we have

$$(1) \qquad \alpha \in \mathcal{R}_z(F) \iff (F)_z\alpha = 0 \ .$$

Let $f^1 = (f_1^1,\dots,f_q^1)$ be the first row of F. Then by hypothesis we can find a neighborhood $U' \subset \Omega$ of 0 and finitely many elements

$$\alpha^{(\nu)} = \begin{pmatrix} \alpha_1^{(\nu)} \\ \vdots \\ \alpha_q^{(\nu)} \end{pmatrix} \in A(U')^q \qquad (\nu = 1,2,\dots,r)$$

so that $(\alpha^{(1)})_z,\dots,(\alpha^{(r)})_z$ generate the A_z-module $\mathcal{R}_z(f^1)$ $(=\mathcal{R}_z(f_1^1,\dots,f_q^1))$ for every $z \in U'$. This means that for any

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} \in A_z{}^r$$

we have

(2) $\quad (f^1)_z \, (A)_z \, c = 0 \qquad (z \in U')$,

where A denotes the matrix with columns $\alpha^{(1)},\dots,\alpha^{(r)}$. Therefore,

(3) $\quad \mathcal{R}_z(F) \subset \{(A)_z \, c \mid c \in A_z\} \qquad (z \in U')$.

We have, because of (1),

(4) $\quad (A)_z \, c \in \mathcal{R}_z(F) \iff (F)_z \, (A)_z \, c = 0$.

But, because of (2), $(F)_z \, (A)_z \, c = 0$ is equivalent to

(5) $\quad (F')_z \, (A)_z \, c = 0 \qquad (z \in U')$,

where F' is the matrix obtained from F by deleting the first row f^1.
In view of (1) now (5) is equivalent to

$$c \in \mathcal{R}_z(F'A) \qquad (z \in U') .$$

So we have

(6) $\quad (A)_z \, c \in \mathcal{R}_z(F) \iff c \in \mathcal{R}_z(F'A) \qquad (z \in U')$.

Since F'A has (p-1) rows it follows by the hypothesis that there is a neighborhood $U'' \subset U'$ of 0 and elements

$$\beta^{(\mu)} = \begin{pmatrix} \beta_1^{(\mu)} \\ \vdots \\ \beta_r^{(\mu)} \end{pmatrix} \in A(U'')^r \qquad (\mu = 1,2,\dots,s) ,$$

so that $(\beta^{(1)})_z, \ldots, (\beta^{(s)})_z$ generate the A_z-module $\mathcal{R}_z(F'A)$ for every $z \in U''$. Let B be the matrix with columns $\beta^{(1)}, \ldots, \beta^{(s)}$. Then any $c \in \mathcal{R}_z(F'A)$ is of the form

(7) $$c = (B)_z d$$

where

$$d = \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix} \in A_z^{\,s} \qquad (z \in U'').$$

From (3), (6) and (7) it follows that the columns of the matrix $(AB)_z$ generate the A_z-module $\mathcal{R}_z(F)$ for every $z \in U''$.

(B)

Assume again that the given point is $z = 0 \in \Omega$. We write $F_i = f_i$ $(1 \leq i \leq q)$ and $f = (f_1, \ldots, f_q)$. After a linear change of coordinates we may suppose that f_i $(1 \leq i \leq q)$ satisfies the conditions of the Weierstrass preparation theorem. Since the assertion of the theorem is local and permits multiplication by units, we may suppose furthermore that

$$f_i = z_n^{\,p_i} + \sum_{\nu=0}^{p_i - 1} a_\nu^{(i)}(z')\, z_n^\nu \qquad (1 \leq i \leq q),$$

where $a_\nu^{(i)}$ is analytic in $\Omega' \subset \mathbb{C}^{n-1}$ $(0 \in \Omega')$ and $a_\nu^{(i)}(0) = 0$.
We may suppose that $p = p_q = \max_{1 \leq i \leq q} p_i$.
Let $\zeta = (\zeta', \zeta_n) \in \Omega$. We say that a relation $(\alpha_1, \ldots, \alpha_q) \in \mathcal{R}_\zeta(f) = \mathcal{R}_\zeta$ is <u>polynomial</u>, if

$$\alpha_i \in A_{n-1,\zeta'}[z_n] \qquad (1 \leq i \leq q).$$

We now prove the following.

(α) Let $\Omega = \Omega' \times D$ where $D = \{z_n \in \mathbb{C} \mid |z_n| < r_n\}$. Then, for any $\zeta = (\zeta', \zeta_n) \in \Omega$, $\mathcal{R}_\zeta$ is generated over $A_{n-1,\zeta'}$ by the polynomial relations in z_n of degree $\leq p$.

Proof of (α).
Write

$$f_q(z',z_n) = u(z)\pi(z',z_n-\zeta_n) ,$$

where π is a Weierstrass polynomial in $z_n - \zeta_n$ with coefficients in $A_{n-1,\zeta'}$ (vanishing at ζ', except for the leading term) of degree $\rho\leq p$, and u is an unit. By lemma 1.1

$$u\in A_{n-1,\zeta'}[z_n]$$

and has degree $p - \rho$ ($\leq p$).
Clearly, for $i>1$, the element

$$s_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_q \\ 0 \\ \vdots \\ 0 \\ -f_i \end{pmatrix} \qquad (f_q \text{ is in the i-th place})$$

is a polynomial relation. If

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} \in \mathcal{R}_\zeta ,$$

we write

$$\alpha_i = c_i\pi + r_i \qquad (1\leq i\leq q-1),$$

where $c_i\in A_{n,\zeta}$ and $r_i\in A_{n-1,\zeta'}[z_n]$ and degree $(r_i) < \rho$. This can be written

$$\alpha_i = d_i f_q + r_i \qquad (1 \leq i \leq q-1),$$

where $d_i = c_i u^{-1} \in A_{n,\zeta}$. Hence

$$(*) \quad \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} - d_1 s_1 - \ldots - d_{q-1} s_{q-1} = \begin{pmatrix} r_1 \\ \vdots \\ r_{q-1} \\ r_q \end{pmatrix} ,$$

where r_q is defined as

$$r_q = \alpha_q + \sum_{i=1}^{q-1} d_i f_i \ .$$

We must have $(r_1, \ldots, r_q) \in \mathcal{R}_\zeta$ since all other terms in the relation $(*)$ are in that module. So we have

$$r_q f_q = - \sum_{i=1}^{q-1} r_i f_i$$

is an element of $A_{n-1,\zeta'}[z_n]$ of degree $<p + \rho$. Hence by lemma 1.1.

$$r_q u \in A_{n-1,\zeta'}[z_n]$$

and has degree $< p$. Also

$$r_i u \in A_{n-1,\zeta'}[z_n] \qquad (1 \leq i \leq q-1)$$

and has degree $< p$. Thus

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} = d_1 s_1 + \ldots + d_{q-1} s_{q-1} + u^{-1} \begin{pmatrix} r_1 u \\ \vdots \\ r_q u \end{pmatrix} .$$

And since $s_1, \ldots, s_{q-1}$ and $(r_1 u, \ldots, r_q u)$ are polynomial relations of degree $\leq p$, the assertion (α) is proved.

To complete the proof of (B) we have therefore only to prove the following:

(β) There exist finitely many polynomial relations

$$\pi^{(\nu)} = \begin{pmatrix} \pi_1^{(\nu)} \\ \vdots \\ \pi_q^{(\nu)} \end{pmatrix}$$

of degree $\leq p$ in a neighborhood U of the origin such that any polynomial relation of degree $\leq p$ at $\zeta \in U$ is generated, over $A_{n-1,\zeta'}$, by the $\pi^{(\nu)}$

Proof of (β).

Let

$$\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_q \end{pmatrix} ,$$

$$\pi_i = \sum_{\nu=0}^{p} c_\nu^{(i)} (z') z_n^\nu \qquad (1 \leq i \leq q)$$

be any polynomial relation at $\zeta = (\zeta', \zeta_n)$ and write

$$f_i = \sum_{\nu=0}^{p} a_\nu^{(i)} (z') z_n^\nu \qquad (1 \leq i \leq q)$$

(note: $a_{p_i}^{(i)} = 1$, $a_\nu^{(i)} = 0$ if $\nu > p_i$).

Then π is a relation if and only if

$$\sum_{i=1}^{q} \sum_{k+l=\nu} a_k^{(i)} (z') c_l^{(i)} (z') = 0 \text{ in } A_{n-1,\zeta'} \quad \text{for } \nu = 0,1,\dots,p.$$

This means that the element

$$(c_0^{(1)},\dots,c_p^{(q)}) \in A_{n-1,\zeta'}^{(p+1)q}$$

is a relation between the (p+1)q sections

$$s_{\nu}^{(i)}(z') = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_0^{(i)}(z') \\ \vdots \\ a_{p-\nu}^{(i)}(z') \end{pmatrix} \in \Gamma(U', \mathcal{A}_{n-1}^{p+1})$$

$(1 \leq i \leq q,\ 0 \leq \nu \leq p)$.

The statement (β) is now an immediate consequence of our induction hypothesis.

3. Analytic Sets.

Definition 3.1. An analytic set (analytic subvariety) in an open set $\Omega \subset \mathbb{C}^n$ is a subset V of Ω with the following property: for every $a \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of a and finitely many analytic functions $f_1, \ldots, f_r$ on U such that

$$V \cap U = \{z \in U \mid f_1(z) = \ldots = f_r(z) = 0\}.$$

Obviously, an analytic set V in Ω is closed in Ω. Furthermore, the interior of V in Ω is both open and closed in Ω. So, if Ω is connected and $V \neq \Omega$, then V is nowhere dense in Ω.

Theorem 3.1. Let V be an analytic set in an open set $\Omega \subset \mathbb{C}^n$. If Ω is connected, then $\Omega \setminus V$ is connected.
Proof. cf. [GR, p. 20].

Theorem 3.2. Let $\{f_i \mid i \in I\}$ be a collection of functions, analytic on an open set Ω. Then

$$\{z \in \Omega \mid f_i(z) = 0 \text{ for all } i \in \mathfrak{I}\}$$

is an analytic set in Ω.
Proof. cf. [GR, p. 86].

Definition 3.2. Let V_1 and V_2 be subsets of an open set $\Omega \subset \mathbb{C}^n$, and let $a \in \Omega$. The sets V_1 and V_2 are said to be equivalent at $a \in \Omega$ (V_1 and V_2 have the same germ at a) if there is a neighborhood U of a such that

$$V_1 \cap U = V_2 \cap U.$$

An equivalence class of this relation is called a germ of a set at a. The germ of a set at a is denoted by $\underline{V}_a$ (or: $\underline{V}$, when no confusion is possible).
The germ of an analytic set is called an analytic germ.

If $\underline{V}_1$ and $\underline{V}_2$ are germs of a set at a, then in an obvious way one defines the germs $\underline{V}_1 \cup \underline{V}_2$ and $\underline{V}_1 \cap \underline{V}_2$; also $\underline{V}_1 \subset \underline{V}_2$ has an obvious meaning.

Definition 3.3. Let $\underline{V}$ be the germ of a set at a; and let $f \in A_{n,a}$ (f is the germ of some function, also denoted by f, which is analytic in a neighborhood U of a). We say that f vanishes on $\underline{V}$ iff $\underline{V}$ is contained in the germ $\underline{V}(f)$ at a of the set

$$V(f) = \{z \mid z \in U,\ f(z) = 0\}.$$

The ideal of $\underline{V}$, $id(\underline{V})$, is defined to be

$$id(\underline{V}) = \{f \mid f \in A_{n,a} \text{ and } f \text{ vanishes on } \underline{V}\} .$$

If $F \subset A_{n,a}$, then the locus of F, loc(F) is defined to be

$$loc(F) = \bigcap_{f \in F} \underline{V}(f)$$

Definition 3.4. If $\mathfrak{J}$ is an ideal in a ring R, we define the radical of $\mathfrak{J}$ to be the ideal

$$Rad(\mathfrak{J}) = \{x \in R \mid \exists\ n \in \mathbb{N} \text{ such that } x^n \in \mathfrak{J}\}.$$

Theorem 3.3. Let $\underline{V}_1$, $\underline{V}_2$ and $\underline{V}$ be analytic germs (at 0), and let $\mathfrak{J}_1$, $\mathfrak{J}_2$ and $\mathfrak{J}$ be ideals in A_n. Then the following relations hold:

(i) $\underline{V}_1 \supset \underline{V}_2$ implies $id(\underline{V}_1) \subset id(\underline{V}_2)$,

(ii) $\mathfrak{J}_1 \supset \mathfrak{J}_2$ implies $loc(\mathfrak{J}_1) \subset loc(\mathfrak{J}_2)$,

(iii) $id(\underline{V}_1) \cap id(\underline{V}_2) = id(\underline{V}_1 \cup \underline{V}_2)$,

(iv) $loc(\mathfrak{J}_1) \cup loc(\mathfrak{J}_2) = loc(\mathfrak{J}_1 \cap \mathfrak{J}_2)$,

(v) $id(\underline{V}) = Rad\ id(\underline{V})$,

(vi) $loc(\mathfrak{J}) = loc(Rad\ \mathfrak{J})$.

Theorem 3.4. Let $\underline{V}$ be an analytic germ, and let $\mathfrak{J}$ be an ideal in A_n. Then

(i) $loc\ id\ (\underline{V}) = \underline{V}$,

(ii) $id\ loc\ (\mathfrak{J}) \supset Rad\ \mathfrak{J}$.

The proofs of Th. 3.3 and Th. 3.4 are simple (cf. [GR, p. 88]).

<u>Definition 3.5</u>. An analytic germ $\underline{V}$ is said to be <u>irreducible</u> if $\underline{V} = \underline{V}_1 \cup \underline{V}_2$, where $\underline{V}_1$ and $\underline{V}_2$ are analytic germs, implies $\underline{V} = \underline{V}_1$ or $\underline{V} = \underline{V}_2$.

<u>Theorem 3.4</u>. An analytic germ $\underline{V}$ is irreducible iff id($\underline{V}$) is prime.
Proof. cf. [GR, p. 89].

<u>Theorem 3.5</u>. Any analytic germ can be written as a finite union

$$\underline{V} = \bigcup_{i=1}^{k} \underline{V}_i$$

of irreducible analytic germs such that

$$\underline{V}_j \not\subset \bigcup_{i \neq j} \underline{V}_i$$

Proof. We need the following theorem:

<u>Theorem</u>. Every ideal $\mathfrak{J}$ in a Noetherian ring is the intersection of finitely many primary ideals $\mathfrak{q}_i$ (i = 1,...,k),

$$\mathfrak{J} = \bigcap_{i=1}^{k} \mathfrak{q}_i$$

where the $\mathfrak{P}_i$ (= Rad $\mathfrak{q}_i$) are unique.
(For a proof, see [W2, §93]).

A decomposition of $\underline{V}$ is obtained as follows:

$$\underline{V} = \text{loc id } (\underline{V}) = \text{loc } \left(\bigcap_{i=1}^{k} \mathfrak{q}_i \right)$$

$$= \bigcup_{i=1}^{k} (\text{loc } \mathfrak{q}_i)$$

$$= \bigcup_{i=1}^{k} (\text{loc } \mathfrak{P}_i)$$

$$= \bigcup_{i=1}^{k} \underline{V}_i \ ;$$

here the $\mathfrak{q}_i$ (i = 1,...,k) are primary, hence the $\mathfrak{P}_i$ = Rad $\mathfrak{q}_i$ (i = 1,...,k) are prime ideals.

Definition 3.6. The germs $\underline{V}_i$ (i = 1,...,k) introduced by the decomposition of Th. 3.5 are called the irreducible components of $\underline{V}$.

Now we come to a deep theorem.

Theorem 3.6. (Hilbert Nullstellensatz for analytic functions).
Let $\mathfrak{J}$ be any ideal of A_n. Then

$$\text{id loc } \mathfrak{J} = \text{Rad } \mathfrak{J}.$$

Proof.
The hard step is to prove the theorem for prime ideals; this requires a detailed investigation of the locus of a prime ideal (an irreducible variety); we refer to [GR, p. 93-97].

Then one easily finishes the proof as follows.
For any ideal $\mathfrak{J}$ of A_n we write:

$$\mathfrak{J} = \bigcap_{i=1}^{k} \mathfrak{Q}_i ,$$

where the ideals $\mathfrak{Q}_i$ are primary, and the ideals Rad $\mathfrak{Q}_i = \mathcal{P}_i$ are prime. The prime ideals $\mathcal{P}_1,\dots,\mathcal{P}_k$ are uniquely determined. Since loc $\mathfrak{Q}_i$ = loc $\mathcal{P}_i$, then

$$\text{loc } \mathfrak{J} = \bigcup_{i=1}^{k} \text{loc } \mathcal{P}_i$$

and

$$\text{id loc } \mathfrak{J} = \bigcap_{i=1}^{k} \text{id loc } \mathcal{P}_i.$$

Since for prime ideals the theorem is supposed to be true, we have

$$\text{id loc } \mathfrak{J} = \bigcap_{i=1}^{k} \mathcal{P}_i = \text{Rad } \mathfrak{J}.$$

Definition 3.7. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. A point $a \in V$ is called a regular point of V of dimension p if there is a neighborhood U of a, $U \subset \Omega$, such that $V \cap U$ is a complex submanifold of dimension p of U. A point $a \in V$ is called singular if it is not regular.

<u>Theorem 3.7</u>. Let V be an analytic set in an open set Ω of $\mathbb{C}^n$. The set of regular points of V is dense in V.
Proof. cf. [GR, p. 111].

<u>Definition 3.8</u>. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. A function f on V is said to be analytic at $a \in V$ if there is a neighborhood U of a in Ω and an analytic function $\tilde{f}$ in U with

$$\tilde{f} \mid U \cap V = f \mid U \cap V.$$

<u>Theorem 3.8</u>. (maximum principle). Let V be an analytic set in Ω and let f be an analytic function on Ω. Let $\underline{V}_a$ be irreducible and suppose that f is not constant on V in any neighborhood of a. Then

$$f(a) \in (f(V))^{\circ}.$$

<u>Corollary</u>. A compact analytic set in $\mathbb{C}^n$ consists of a finite number of points.
Proof. cf. [GR, p. 104-106].

<u>Definition 3.9</u>. Let V be an analytic set in an open set Ω in $\mathbb{C}^n$. For $z \in \Omega$, let $\mathfrak{J}_z(V)$ the ideal $\mathrm{id}(\underline{V}_z)$ of A_z (the ideal of A_z of all germs of analytic functions vanishing on $\underline{V}_z$) (if $z \notin V$, then $\mathfrak{J}_z(V) = A_z$). Then

$$\mathfrak{J}(V) = \bigcup_{z \in \Omega} \mathfrak{J}_z(V)$$

defines a subsheaf of $\mathcal{A}(\Omega)$ (= $\mathcal{A}_{n,\Omega}$) (note that $\mathfrak{J}(V)$ is an open subset of $\mathcal{A}(\Omega)$). $\mathfrak{J}(V)$ is called the <u>sheaf of ideals</u> of the analytic set V.

<u>Theorem 3.9</u>. If V is an analytic set in an open set $\Omega \subset \mathbb{C}^n$, then $\mathfrak{J}(V)$ is a coherent analytic sheaf on Ω
Proof. cf. [GR, p. 138-141].

From this coherence theorem one can deduce the following interesting fact.

Theorem 3.10. Let V be an analytic set in an open set $\Omega \subset \mathbb{C}^n$. Then the set of singular points of V is again an analytic set in Ω.

Corollary. Any analytic set in an open set in $\mathbb{C}^n$ can be written as an union of complex manifolds.

Proof. cf. [GR, p. 141-142].

Let V be an analytic set in an open set $\Omega \subset \mathbb{C}^n$. Let us introduce the quotientsheaf

$$\tilde{\mathcal{H}} = \mathcal{A}(\Omega) / \mathcal{J}(V)$$

(observe that $\tilde{\mathcal{H}}_z = 0$ if $z \in \Omega \setminus V$).

Let $\mathcal{H}$ be the restriction of $\tilde{\mathcal{H}}$ to V. Then $\mathcal{H}$ can be identified with the sheaf of germs of analytic functions on V.

We now state special cases of Cartan's Theorem A and B.

Theorem 3.11. Let Ω be a domain of holomorphy in C^n and let $\mathcal{F}$ be a coherent analytic sheaf on Ω. Then the following hold:

A. $\Gamma(\Omega, \mathcal{F})$ generates $\mathcal{F}_z$ for all $z \in \Omega$,

B. $H^1(\Omega, \mathcal{F}) = 0$.

We shall give two applications of this theorem.

Theorem 3.12. Let Ω be a domain of holomorphy and let V be an analytic set in Ω. Then

$$V = \{z \in \Omega \mid f_i(z) = 0 \ \forall\, i \in I\} ,$$

where the f_i are analytic in Ω and I is some index set.

(This theorem says that a local simultaneous zero set).

Proof. The sheaf $\mathcal{J}(V)$ on Ω is coherent (Th. 3.9).

Hence (Th. 3.11. A) the global sections of $\mathcal{J}(V)$ generate the stalk of $\mathcal{J}(V)$ at each point $z \in \Omega$. For $z \in \Omega \setminus V$, the stalk $\mathcal{J}_z(V)$ contains the germ 1_z. Hence there are $f_j \in \Gamma(\Omega, \mathcal{J}(V))$ and $g_j \in A_z$ $(j = 1,\dots,k)$ such that

$$1_z = \sum g_j (f_j)_z .$$

Thus some $f_j(z) \neq 0$.

<u>Theorem 3.13</u>. Let Ω be a domain of holomorphy and let V be an analytic set in Ω. Then every function analytic on V is the restriction of a function analytic on Ω.

Proof. We have the following exact sequence of sheaves:

$$0 \to \mathfrak{I}(V) \to \mathcal{A}(\Omega) \to \tilde{\mathcal{H}} \to 0 \ ;$$

and so the exact cohomology sequence:

$$\ldots \to \Gamma(\Omega, \mathcal{A}) \to \Gamma(\Omega, \tilde{\mathcal{H}}) \to H^1(\Omega, \mathfrak{I}(V)) \to \ldots$$

Now $\Gamma(\Omega, \tilde{\mathcal{H}}) = \Gamma(V, \mathcal{H})$.

Furthermore (Th. 3.11.B): $H^1(\Omega, \mathfrak{I}(V)) = 0$.

So the theorem follows.

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